This article is written by students. It may include omissions or imperfections, reported as far as possible by our reviewers in the editing notes

The roof is on fire

[Year 2020-2021]

Surnames and first names of students, grades: GALL Ștefania, GRUMĂZESCU George-Octavian, LEFTER Lucia-Maria, MOCANU Radu, OANȚĂ Ana-Estera, RUSU George-Constantin, students in 11th grade

School: Colegiul Național “Costache Negruzzi”, Iași

Teacher: Adrian ZANOSCHI

Reasearcher: Iulian STOLERIU, Faculty of Mathematics, “Alexandru Ioan Cuza” University of Iași

1 PRESENTATION OF THE RESEARCH TOPIC

Problems that require determining the optimal trajectory between two points under certain restrictions often occur in practice. In Section 2 of this paper, we try to find the position of a point P such that the path that joins two given points, passing through P, is traveled in minimum time. The speeds with which the road is traveled until the arrival in P and after leaving P are different. In Section 3 we consider the speed constant along the trajectory, but we impose more restrictions on the trajectory.

2 FASTEST PATH PROBLEMS

Problem 1. A tourist is camping on a shore of a straight river. At a given moment, he finds himself at position A (Figure 1), when he sees that the roof of his camping house (situated at position B, on the same side of the river), is on fire. He quickly grabs an empty bucket and wishes to put down the fire with water taken from the river. He runs twice as fast with an empty bucket as with a full one. Where should he get the water along the river to minimize the total travel time to the house?

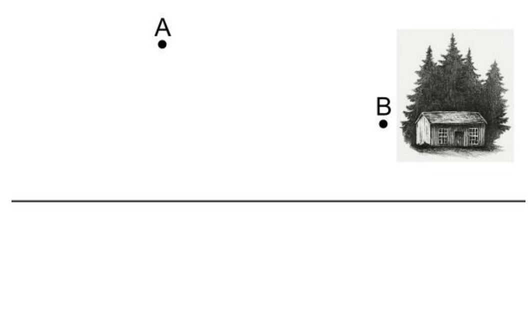

Figure 1

Solution. Let us denote by s the line where the shore touches the river and by P the point on the shore of the river where the tourist fills the bucket. Taking into account that the man runs twice as fast with an empty bucket as with a full one, let $2v$ be the speed of the tourist with an empty bucket and v his speed with a full bucket. The time in which the man covers the distance $AP+PB$ will be therefore

$$t=\frac{AP}{2v}+\frac{PB}{v}=\frac{1}{2v}\left(AP+2PB\right).$$

Since v is a constant, it results that t will be minimal if and only if the distance $AP+2PB$ is minimal.

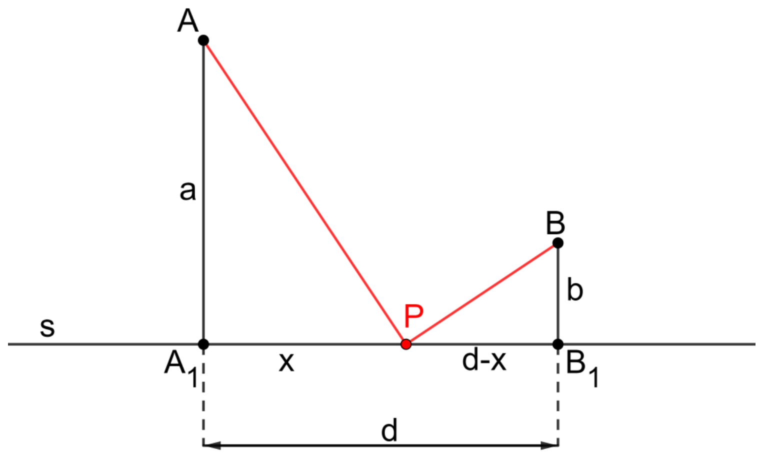

Figure 2

Let A_1 and B_1 be the projections of A and B, respectively, onto the line s of the shore; we expect that $P\in\left[A_1B_1\right]$. We denote $AA_1=a>0$, $BB_1=b>0$, $A_1B_1=d>0$. If $A_1P=x\in\left[0,d\right]$, then $PB_1=A_1B_1-A_1P=d-x$. By the Pythagorean Theorem,

$$AP=\sqrt{a^2+x^2},\quad PB=\sqrt{b^2+\left(d-x\right)^2},$$

so, the problem reduces to finding the minimum of the function $f:[0,d]\to\mathbb{R}$,

$$f\left(x\right)=\sqrt{a^2+x^2}+2\sqrt{b^2+\left(d-x\right)^2}.$$

Being obtained by compositions and operations with elementary functions, the function f is differentiable and its derivative is

$$f'(x)=\frac{1}{2\sqrt{a^2+x^2}}\cdot\left(a^2+x^2\right)'+2\cdot\frac{1}{2\sqrt{b^2+(d-x)^2}}\cdot\left(b^2+(d-x)^2\right)'$$
$$=\frac{2x}{2\sqrt{a^2+x^2}}-2\cdot\frac{2(d-x)}{2\sqrt{b^2+(d-x)^2}}=\frac{x}{\sqrt{a^2+x^2}}-\frac{2(d-x)}{\sqrt{b^2+(d-x)^2}},$$

for all $x\in[0,d]$. At its turn, f' is also differentiable and its derivative is

$$f''(x)=\frac{\sqrt{a^2+x^2}-x\cdot\frac{x}{\sqrt{a^2+x^2}}}{a^2+x^2}-\frac{-2\sqrt{b^2+(d-x)^2}-2(d-x)\cdot\frac{-(d-x)}{\sqrt{b^2+(d-x)^2}}}{b^2+(d-x)^2}$$
$$=\frac{a^2+x^2-x^2}{\left(\sqrt{a^2+x^2}\right)^3}-\frac{-2\left[b^2+(d-x)^2\right]+2(d-x)^2}{\left(\sqrt{b^2+(d-x)^2}\right)^3}$$
$$=\frac{a^2}{\left(\sqrt{a^2+x^2}\right)^3}+\frac{2b^2}{\left(\sqrt{b^2+(d-x)^2}\right)^3},$$

for all $x\in[0,d]$. It is obvious that $f''(x)>0$ for all $x\in[0,d]$, which implies that f' is a strictly increasing function on $[0,d]$. Since f' is a continuous function and $f'(0)=-\frac{2d}{\sqrt{b^2+d^2}}<0$, while $f'(d)=\frac{d}{\sqrt{a^2+d^2}}>0$, it results that there exists a unique point $x_0\in(0,d)$ such that $f'(x_0)=0$.

Moreover, since f' is strictly increasing and $f'(x_0)=0$, it follows that $f'<0$ on $[0,x_0)$ (hence, f is strictly decreasing on $[0,x_0)$) and $f'>0$ on $(x_0,d]$ (hence f is strictly increasing on $(x_0,d]$). Consequently, f attains its minimal value in x_0 (and only in x_0).

x	0		x_0		d
$f''(x)$	$+ \; + \; +$		$+ \; + \; +$		$+ \; + \; +$
$f'(x)$	$\nearrow \; \nearrow$ $- \; -$		0		$\nearrow \; \nearrow$ $+ \; +$
$f(x)$	$\searrow \; \searrow$		$f(x_0)$ min		$\nearrow \; \nearrow$

The point x_0 is the unique solution of the equation $f'(x)=0$, which is

$$\frac{x}{\sqrt{a^2+x^2}}-\frac{2(d-x)}{\sqrt{b^2+(d-x)^2}}=0 \Leftrightarrow \frac{x}{\sqrt{a^2+x^2}}=\frac{2(d-x)}{\sqrt{b^2+(d-x)^2}} \Leftrightarrow \frac{x^2}{a^2+x^2}=\frac{4(d-x)^2}{b^2+(d-x)^2}$$

$$\Leftrightarrow x^2\left[b^2+(d-x)^2\right]=4(d-x)^2\left(a^2+x^2\right) \Leftrightarrow b^2x^2+x^2(d-x)^2=4(d-x)^2a^2+4(d-x)^2x^2$$

$$\Leftrightarrow b^2x^2-4(d-x)^2a^2=3(d-x)^2x^2 \Leftrightarrow \frac{b^2}{(d-x)^2}-\frac{4a^2}{x^2}=3 \Leftrightarrow \left(\frac{b}{d-x}\right)^2-\left(\frac{2a}{x}\right)^2=3.$$

Particular cases

(i) Since $3=2^2-1^2$, we could have $b=2(d-x)$ and $x=2a$, which implies $b=2d-4a$. In other words, whenever $d>2a$ and $(AA_1, BB_1, A_1B_1)=(a, 2d-4a, d)$, the minimum point lays at distance $x=2a$ from A_1.

Examples of such triplets:

$(a,a,2.5a);(a,2a,3a),(a,4a,4a),(a,6a,5a),(a,8a,6a),(a,10a,7a),(a,12a,8a),\ldots$, in general $(a,2(k-2)a,ka)$, with $k>2$ ($a>0$).

(ii) Since $3^2\cdot 3=14^2-13^2$, which is equivalent to $\left(\frac{14}{3}\right)^2-\left(\frac{13}{3}\right)^2=3$, we could have $b=\frac{14}{3}(d-x)$ and $2a=\frac{13}{3}x \Leftrightarrow x=\frac{6a}{13}$, which implies $b=14\left(\frac{d}{3}-\frac{2a}{13}\right)$. In other words, whenever $d>\frac{6a}{13}$ and $\left(AA_1, BB_1, A_1B_1\right)=\left(a, 14\left(\frac{d}{3}-\frac{2a}{13}\right), d\right)$, the minimum point lays at distance $x=\frac{6a}{13}$ from A_1. Examples: $(13s,14s,9s),(13s,28s,12s)$, with $s>0$ imply $x=6s$.

(iii) Since $4^2\cdot 3=7^2-1^2$, which is equivalent to $\left(\frac{7}{4}\right)^2-\left(\frac{1}{4}\right)^2=3$, we could have $b=\frac{7}{4}(d-x)$ and $2a=\frac{x}{4}\Leftrightarrow x=8a$, which implies $b=\frac{7}{4}(d-8a)$. In other words, whenever $d>8a$ and $\left(AA_1, BB_1, A_1B_1\right)=\left(a, \frac{7}{4}(d-8a), d\right)$, the minimum point lays at distance $x=8a$ from A_1. Examples of such triplets are: $(a,7a,12a),(a,14a,16a)$, with $a>0$.

(iv) Since $4^2\cdot 3=13^2-11^2$, which is equivalent to $\left(\frac{13}{4}\right)^2-\left(\frac{11}{4}\right)^2=3$, we could have $b=\frac{13}{4}(d-x)$ and $2a=\frac{11x}{4}\Leftrightarrow x=\frac{8a}{11}$, which implies $b=\frac{13}{4}\left(d-\frac{8a}{11}\right)$. In other words, whenever $d>\frac{8a}{11}$ and $\left(AA_1, BB_1, A_1B_1\right)=\left(a, \frac{13}{4}\left(d-\frac{8a}{11}\right), d\right)$, the minimum point lays at distance $x=\frac{8a}{11}$ from A_1. Examples: $(11s,13s,12s),(11s,26s,16s)$, with $s>0$ imply $x=8s$.

The first computer program reads the lengths `a,b,d` of segments AA_1, BB_1, A_1B_1 and the step `pas` and makes the point P move on the segment $[A_1B_1]$from A_1 to B_1 with step `pas`, computing the distance `dist`$=AP+2PB$ and determining (approximately) the position of point P for which this distance is minimal. [1]

```cpp
#include <iostream>
#include <math.h>
using namespace std;
double a,b,d,x,pas,minim,xmin,dist;
int main()
{
    cout<<"AA1=";
    cin>>a;
    cout<<"BB1=";
    cin>>b;
    cout<<"A1B1=";
    cin>>d;
    cout<<"Step=";
    cin>>pas;
    minim=1.7976931348623158E+308-1;
    while(x<=d)
    {
        dist=sqrt(a*a+x*x)+2*sqrt(b*b+(d-x)*(d-x));
        if(minim>dist)
        {
            minim=dist;
            xmin=x;
        }
        cout<<"A1P="<<x<<"  dist="<<dist<<endl;
        x=x+pas;
    }
    cout<<endl<<"A1P_min="<<xmin<<"; minimal distance="<<minim<<endl;
    return 0;
}
```

```
AA1=1
BB1=2
A1B1=3
Step=0.1
A1P=0  dist=8.2111
A1P=0.1  dist=8.05055
A1P=0.2  dist=7.90166
A1P=0.3  dist=7.76415
A1P=0.4  dist=7.63752
A1P=0.5  dist=7.52116
A1P=0.6  dist=7.41439
A1P=0.7  dist=7.31656
A1P=0.8  dist=7.22705
A1P=0.9  dist=7.14536
A1P=1  dist=7.07107
A1P=1.1  dist=7.00385
A1P=1.2  dist=6.9435
A1P=1.3  dist=6.88988
A1P=1.4  dist=6.84296
A1P=1.5  dist=6.80278
A1P=1.6  dist=6.76942
A1P=1.7  dist=6.74305
A1P=1.8  dist=6.72389
A1P=1.9  dist=6.71218
A1P=2  dist=6.7082
A1P=2.1  dist=6.71228
A1P=2.2  dist=6.72474
A1P=2.3  dist=6.74591
A1P=2.4  dist=6.77612
A1P=2.5  dist=6.81569
A1P=2.6  dist=6.86489
A1P=2.7  dist=6.92399
A1P=2.8  dist=6.99316
A1P=2.9  dist=7.07257
A1P=3  dist=7.16228

A1P_min=2; minimal distance=6.7082
```

The second program makes the lengths `a,b,d` of segments AA_1, BB_1, A_1B_1 vary from zero (exclusive) to a maximum value (`maxa, maxb, maxd`, respectively) with step `pa, pb, pd`, respectively, where `pa, pb, pd, maxa, maxb, maxd` and the step `pas` for the point P on the segment $[A_1B_1]$ are entered by the user. It computes the minimum of `dist` $= AP + 2PB$, displaying it along with the position of P on $[A_1B_1]$.

This allows us to identify particular cases of `a,b,d` for which the distance $x = A_1P$ has a simpler form.

main.cpp

```cpp
#include <bits/stdc++.h>
using namespace std;
double a,b,d,x,pas,minim,xmin,dist;
double pa,pb,pd,maxa,maxb,maxd;
long long i;
int main()
{
    cout<<"The step on AA1=";
    cin>>pa;
    cout<<"The step on BB1=";
    cin>>pb;
    cout<<"The step on A1B1=";
    cin>>pd;
    cout<<"Maximum value for a=";
    cin>>maxa;
    cout<<"Maximum value for b=";
    cin>>maxb;
    cout<<"Maximum value for d=";
    cin>>maxd;
    cout<<"The step on d=";
    cin>>pas;
```

C/C++ | Windows (CR+LF) | WINDOWS-1252 | Line 30, Col 14, Pos 632 | Insert | Read/Write | defa

main.cpp

```cpp
    i=0;a=pa;
    while(a<=maxa)
    {
        b=pb;
        while(b<=maxb)
        {
            d=pd;
            while(d<=maxd)
            {
                x=0; minim=1.7976931348623158E+308-1;
                while(x<=d)
                {
                    dist=sqrt(a*a+x*x)+2*sqrt(b*b+(d-x)*(d-x));
                    if(minim>dist)
                    {
                        minim=dist;xmin=x;
                    }
                    //cout<<"A1P="<<x<<"  dist="<<dist<<endl;
                    x=x+pas;
                }
                i++;
                cout<<i<<") a="<<a<<"; b="<<b<<"; d="<<d<<";  minimal dist ="<<minim<<"; A1P="<<xmin<<endl;
                d=d+pd;}
            b=b+pb;}
        a=a+pa;}
    return 0;
}
```

```
The step on AA1=1
The step on BB1=1
The step on A1B1=1
Maximum value for a=5
Maximum value for b=5
Maximum value for d=5
The step on d=0.001
1) a=1; b=1; d=1;  minimal dist =3.30872; A1P=0.701
2) a=1; b=1; d=2;  minimal dist =4.03764; A1P=1.538
3) a=1; b=1; d=3;  minimal dist =4.92826; A1P=2.477
4) a=1; b=1; d=4;  minimal dist =5.87454; A1P=3.452
5) a=1; b=1; d=5;  minimal dist =6.84347; A1P=4.441
6) a=1; b=2; d=1;  minimal dist =5.2407; A1P=0.523
7) a=1; b=2; d=2;  minimal dist =5.86988; A1P=1.176
8) a=1; b=2; d=3;  minimal dist =6.7082; A1P=2
9) a=1; b=2; d=4;  minimal dist =7.63242; A1P=2.926
10) a=1; b=2; d=5;  minimal dist =8.59123; A1P=3.893
11) a=1; b=3; d=1;  minimal dist =7.1957; A1P=0.415
12) a=1; b=3; d=2;  minimal dist =7.73578; A1P=0.92
13) a=1; b=3; d=3;  minimal dist =8.50797; A1P=1.597
14) a=1; b=3; d=4;  minimal dist =9.39968; A1P=2.435
15) a=1; b=3; d=5;  minimal dist =10.3437; A1P=3.362
16) a=1; b=4; d=1;  minimal dist =9.16438; A1P=0.343
17) a=1; b=4; d=2;  minimal dist =9.63152; A1P=0.746
18) a=1; b=4; d=3;  minimal dist =10.3326; A1P=1.283
19) a=1; b=4; d=4;  minimal dist =11.1803; A1P=2
20) a=1; b=4; d=5;  minimal dist =12.1027; A1P=2.859
21) a=1; b=5; d=1;  minimal dist =11.1415; A1P=0.292
22) a=1; b=5; d=2;  minimal dist =11.5505; A1P=0.625
23) a=1; b=5; d=3;  minimal dist =12.1836; A1P=1.054
24) a=1; b=5; d=4;  minimal dist =12.9788; A1P=1.64
25) a=1; b=5; d=5;  minimal dist =13.8712; A1P=2.399
26) a=2; b=1; d=1;  minimal dist =4.19358; A1P=0.809
27) a=2; b=1; d=2;  minimal dist =4.71159; A1P=1.663
28) a=2; b=1; d=3;  minimal dist =5.43358; A1P=2.571
29) a=2; b=1; d=4;  minimal dist =6.26697; A1P=3.517
30) a=2; b=1; d=5;  minimal dist =7.16037; A1P=4.487
31) a=2; b=2; d=1;  minimal dist =6.1633; A1P=0.676
32) a=2; b=2; d=2;  minimal dist =6.61743; A1P=1.401
33) a=2; b=2; d=3;  minimal dist =7.28134; A1P=2.203
34) a=2; b=2; d=4;  minimal dist =8.07529; A1P=3.077
35) a=2; b=2; d=5;  minimal dist =8.94427; A1P=4
36) a=2; b=3; d=1;  minimal dist =8.14092; A1P=0.579
37) a=2; b=3; d=2;  minimal dist =8.54205; A1P=1.201
38) a=2; b=3; d=3;  minimal dist =9.14924; A1P=1.9
39) a=2; b=3; d=4;  minimal dist =9.89912; A1P=2.687
40) a=2; b=3; d=5;  minimal dist =10.7387; A1P=3.548
41) a=2; b=4; d=1;  minimal dist =10.1238; A1P=0.506
42) a=2; b=4; d=2;  minimal dist =10.4814; A1P=1.047
43) a=2; b=4; d=3;  minimal dist =11.0361; A1P=1.655
44) a=2; b=4; d=4;  minimal dist =11.7398; A1P=2.352
45) a=2; b=4; d=5;  minimal dist =12.5455; A1P=3.14
46) a=2; b=5; d=1;  minimal dist =12.1103; A1P=0.449
47) a=2; b=5; d=2;  minimal dist =12.4321; A1P=0.926
48) a=2; b=5; d=3;  minimal dist =12.9398; A1P=1.459
49) a=2; b=5; d=4;  minimal dist =13.5975; A1P=2.072
50) a=2; b=5; d=5;  minimal dist =14.366; A1P=2.78
51) a=3; b=1; d=1;  minimal dist =5.14034; A1P=0.861
52) a=3; b=1; d=2;  minimal dist =5.53458; A1P=1.741
53) a=3; b=1; d=3;  minimal dist =6.12177; A1P=2.649
54) a=3; b=1; d=4;  minimal dist =6.84002; A1P=3.585
55) a=3; b=1; d=5;  minimal dist =7.64311; A1P=4.541
56) a=3; b=2; d=1;  minimal dist =7.12345; A1P=0.754
57) a=3; b=2; d=2;  minimal dist =7.47659; A1P=1.533
58) a=3; b=2; d=3;  minimal dist =8.01679; A1P=2.351
59) a=3; b=2; d=4;  minimal dist =8.69438; A1P=3.215
60) a=3; b=2; d=5;  minimal dist =9.46658; A1P=4.117
61) a=3; b=3; d=1;  minimal dist =9.1101; A1P=0.671
62) a=3; b=3; d=2;  minimal dist =9.42888; A1P=1.365
63) a=3; b=3; d=3;  minimal dist =9.92615; A1P=2.102
64) a=3; b=3; d=4;  minimal dist =10.5631; A1P=2.89
65) a=3; b=3; d=5;  minimal dist =11.3022; A1P=3.731
66) a=3; b=4; d=1;  minimal dist =11.0993; A1P=0.604
67) a=3; b=4; d=2;  minimal dist =11.3892; A1P=1.228
68) a=3; b=4; d=3;  minimal dist =11.848; A1P=1.893
69) a=3; b=4; d=4;  minimal dist =12.4457; A1P=2.61
70) a=3; b=4; d=5;  minimal dist =13.1505; A1P=3.386
71) a=3; b=5; d=1;  minimal dist =13.0904; A1P=0.549
72) a=3; b=5; d=2;  minimal dist =13.3559; A1P=1.115
73) a=3; b=5; d=3;  minimal dist =13.7806; A1P=1.718
74) a=3; b=5; d=4;  minimal dist =14.3412; A1P=2.37
75) a=3; b=5; d=5;  minimal dist =15.0115; A1P=3.081
76) a=4; b=1; d=1;  minimal dist =6.10988; A1P=0.891
77) a=4; b=1; d=2;  minimal dist =6.42587; A1P=1.791
78) a=4; b=1; d=3;  minimal dist =6.91397; A1P=2.708
79) a=4; b=1; d=4;  minimal dist =7.53394; A1P=3.642
80) a=4; b=1; d=5;  minimal dist =8.24992; A1P=4.593
81) a=4; b=2; d=1;  minimal dist =8.09916; A1P=0.802
82) a=4; b=2; d=2;  minimal dist =8.38716; A1P=1.618
83) a=4; b=2; d=3;  minimal dist =8.83915; A1P=2.458
84) a=4; b=2; d=4;  minimal dist =9.42317; A1P=3.325
85) a=4; b=2; d=5;  minimal dist =10.1079; A1P=4.221
86) a=4; b=3; d=1;  minimal dist =10.0903; A1P=0.73
87) a=4; b=3; d=2;  minimal dist =10.3545; A1P=1.474
88) a=4; b=3; d=3;  minimal dist =10.774; A1P=2.243
89) a=4; b=3; d=4;  minimal dist =11.3238; A1P=3.046
90) a=4; b=3; d=5;  minimal dist =11.9771; A1P=3.885
91) a=4; b=4; d=1;  minimal dist =12.0829; A1P=0.669
92) a=4; b=4; d=2;  minimal dist =12.3266; A1P=1.351
93) a=4; b=4; d=3;  minimal dist =12.7172; A1P=2.06
94) a=4; b=4; d=4;  minimal dist =13.2349; A1P=2.802
95) a=4; b=4; d=5;  minimal dist =13.8572; A1P=3.584
96) a=4; b=5; d=1;  minimal dist =14.0766; A1P=0.617
97) a=4; b=5; d=2;  minimal dist =14.3026; A1P=1.247
98) a=4; b=5; d=3;  minimal dist =14.6675; A1P=1.901
99) a=4; b=5; d=4;  minimal dist =15.1553; A1P=2.589
100) a=4; b=5; d=5;  minimal dist =15.7477; A1P=3.316
101) a=5; b=1; d=1;  minimal dist =7.09022; A1P=0.91
102) a=5; b=1; d=2;  minimal dist =7.35305; A1P=1.826
103) a=5; b=1; d=3;  minimal dist =7.76791; A1P=2.752
104) a=5; b=1; d=4;  minimal dist =8.30808; A1P=3.689
105) a=5; b=1; d=5;  minimal dist =8.94691; A1P=4.638
106) a=5; b=2; d=1;  minimal dist =9.08283; A1P=0.835
107) a=5; b=2; d=2;  minimal dist =9.32557; A1P=1.678
108) a=5; b=2; d=3;  minimal dist =9.7126; A1P=2.536
109) a=5; b=2; d=4;  minimal dist =10.2225; A1P=3.412
110) a=5; b=2; d=5;  minimal dist =10.8326; A1P=4.309
111) a=5; b=3; d=1;  minimal dist =11.0765; A1P=0.771
112) a=5; b=3; d=2;  minimal dist =11.3019; A1P=1.551
113) a=5; b=3; d=3;  minimal dist =11.6639; A1P=2.348
114) a=5; b=3; d=4;  minimal dist =12.1456; A1P=3.167
115) a=5; b=3; d=5;  minimal dist =12.7276; A1P=4.012
116) a=5; b=4; d=1;  minimal dist =13.0711; A1P=0.716
117) a=5; b=4; d=2;  minimal dist =13.2812; A1P=1.441
118) a=5; b=4; d=3;  minimal dist =13.6209; A1P=2.183
119) a=5; b=4; d=4;  minimal dist =14.0764; A1P=2.949
120) a=5; b=4; d=5;  minimal dist =14.6315; A1P=3.744
121) a=5; b=5; d=1;  minimal dist =15.0664; A1P=0.668
122) a=5; b=5; d=2;  minimal dist =15.2632; A1P=1.345
123) a=5; b=5; d=3;  minimal dist =15.5828; A1P=2.039
124) a=5; b=5; d=4;  minimal dist =16.0141; A1P=2.756
125) a=5; b=5; d=5;  minimal dist =16.5436; A1P=3.503

Process returned 0 (0x0)   execution time : 18.081 s
Press any key to continue.
```

Remark. We can use a similar argument to try to locate the point P if the speed of the tourist with an empty bucket is k times his speed with a full bucket, where $k>0$ is a constant. This time we have to minimize $AP+kPB$. With the notations in Figure 2, the problem reduces to finding the minimum of the function $f:[0,d]\longrightarrow\mathbb{R}$

$$f(x)=\sqrt{a^2+x^2}+k\sqrt{b^2+(d-x)^2},$$

which is differentiable and has the derivative

$$f'(x)=\frac{x}{\sqrt{a^2+x^2}}-\frac{k(d-x)}{\sqrt{b^2+(d-x)^2}},$$

for all $x\in[0,d]$. Also using the second derivative, we get that f' is strictly increasing on $[0,d]$ and since it is obviously continuous, while $f'(0)\,f'(d)=-\frac{kd}{\sqrt{b^2+d^2}}\cdot\frac{d}{\sqrt{a^2+d^2}}<0$, we infer that there exists a unique point $x_0\in(0,d)$ such that $f'(x_0)=0$. Proceeding as above, we conclude that f attains its minimal value in x_0 (and only in x_0).

The point x_0 verifies the equation $f'(x)=0$, which is

$$\frac{x}{\sqrt{a^2+x^2}}-\frac{k(d-x)}{\sqrt{b^2+(d-x)^2}}=0\Leftrightarrow x^2\left[b^2+(d-x)^2\right]=k^2(d-x)^2\left(a^2+x^2\right)$$

$$\Leftrightarrow b^2x^2-k^2(d-x)^2a^2=\left(k^2-1\right)(d-x)^2x^2\Leftrightarrow\left(\frac{b}{d-x}\right)^2-\left(\frac{ka}{x}\right)^2=k^2-1.$$

Problem 2. The same question as in Problem 1, but this time the house is situated at point C, on the other side of the river. Suppose that the swimming speed equals the running speed with a full bucket and the river water is calm.

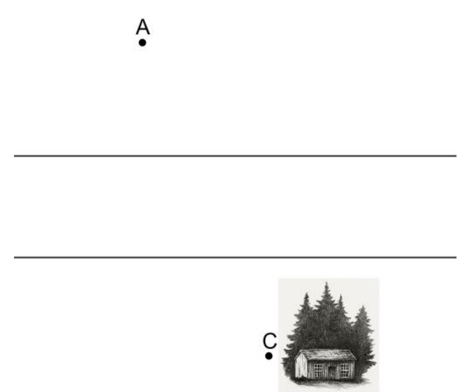

Figure 3

Solution. Let P_1 be the point of the shore s where the tourist jumps into the river and begins to swim. If his swimming speed equals his running speed with a full bucket and the river water is calm, then the man saves the most time if the trajectory P_1C is a straight line. Let the width of the river be $w > 0$. Let C_2, C_1 be the projections of C onto the lines s' and s of the shores of the river (thus $C_1C_2 = w$); we expect that $P_1 \in [A_1C_1]$. We denote $AA_1 = a > 0$, $CC_2 = c > 0$, $A_1C_1 = d_1 > 0$. If $A_1P_1 = x \in [0, d_1]$, then $P_1C_1 = A_1C_1 - A_1P_1 = d_1 - x$. By the Pythagorean Theorem,

$$AP_1 = \sqrt{a^2 + x^2}, \quad P_1C = \sqrt{(w+c)^2 + (d_1 - x)^2},$$

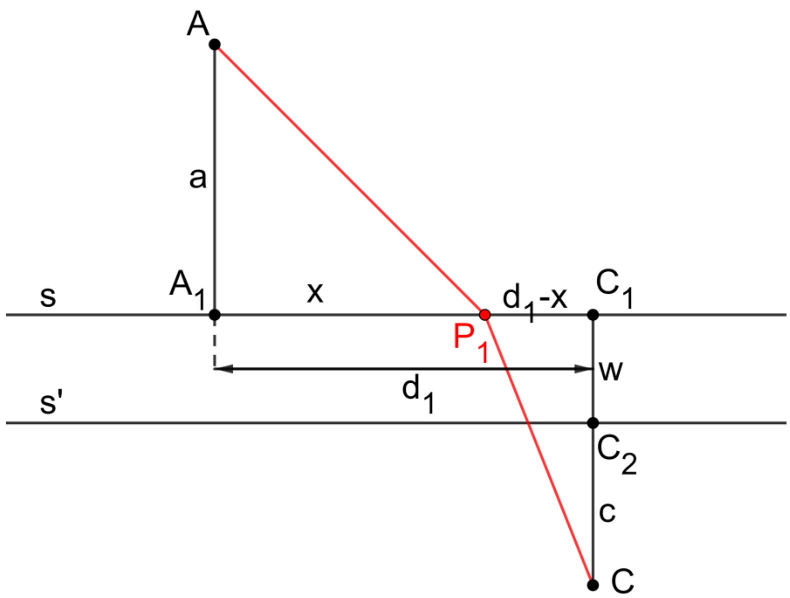

Figure 4

so, the problem reduces to finding the minimum of the function $g: [0, d_1] \to \mathbb{R}$,

$$g(x) = \sqrt{a^2 + x^2} + 2\sqrt{(w+c)^2 + (d_1 - x)^2}.$$

This problem is similar with Problem 1, the segment $[BB_1]$ being replaced by $[CC_1]$. Therefore, the solution of Problem 2 can be obtained from that of Problem 1 by replacing the distance b from point B to the river with the sum $w + c$ (the width of the river summed with the distance from point C to the river).

3 SHORTEST PATH PROBLEMS

Starting from the research topic formulated at the beginning of the article, we set out to study some similar problems. The difference between Problems 1 and 2 above and the problems below is that we assume that *the tourist maintains the same speed.* Therefore, *he will travel the road in minimum time if and only if he chooses the shortest path*. This makes Problem 3 below simpler than Problem 1. Also, this hypothesis allows us to use a completely different approach, namely a geometrical one.

Problem 4 is slightly more complicated than Problem 3, because we need to find a minimal trajectory that touches two intersecting lines. Finally, in Problem 5 we find the position of bridges crossing one or more channels which ensures a minimal path.

Problem 3. If the speed of the tourist with an empty bucket or with a full bucket is the same, where should he get the water along the river to minimize the total travel time to the house?

Solution. Let us suppose that the tourist has the same speed v with an empty or a full bucket. If s is the line where the shore touches the river and P is a point on the shore of the river, then the time in which the man covers the distance $AP+PB$ will be

$$t=\frac{AP}{v}+\frac{PB}{v}=\frac{1}{v}\left(AP+PB\right).$$

Since v is a constant, it results that t will be minimal if and only if the distance $AP+PB$ is minimal.

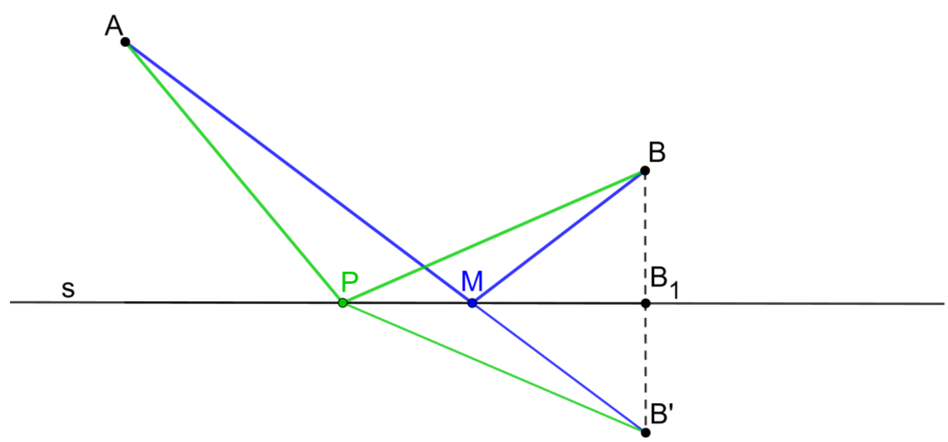

Figure 5

Let B_1 and B' be the projection of B onto s and the symmetrical of B with respect to the line s, respectively (see Figure 5). Let $s \cap [AB'] = \{M\}$. Since s is the mediator of the segment $[BB']$, both triangles MBB' and PBB' are isosceles triangles, therefore $MB = MB'$ and $PB = PB'$. It results that $AM + MB = AM + MB' = AB'$ and $AP + PB = AP + PB'$. But for all $P \neq M$ the triangular inequality in the triangle APB' implies that $AP + PB' > AB' = AM + MB$. In conclusion, the point on the shore where the tourist should fill his bucket is M.

Problem 4. Let A and B be two points located on an island at the confluence of two rivers (see Figure 6). John participates to the following photography contest: he has to go from A to the bank of the first river and take a picture, then on the bank of the second river and take another picture and finish his route in B. Where should he reach the banks of the two rivers in order to cover a minimal distance?

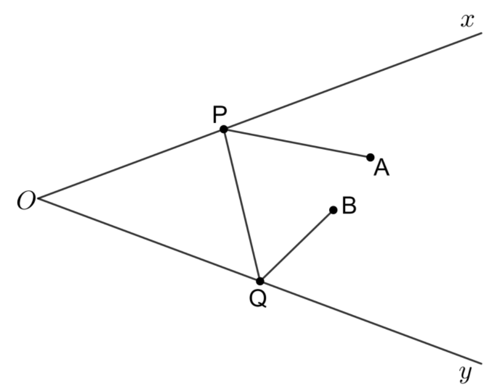

Figure 6

Solution. Let A_1 and A' be the projection of A onto Ox and the symmetrical of A with respect to the line Ox, respectively Let B_1 and B' be the projection of B onto Oy and the symmetrical of B with respect to the line Oy, respectively (see Figure 7). Let $[A'B'] \cap Ox = \{M\}$ and $[A'B'] \cap Oy = \{N\}$. Let $P \in [Ox$ and $Q \in [Oy$ such that $P \neq M$ or $Q \neq N$. Since $[Ox$ is the mediator of the segment $[AA']$, both triangles MAA' and PAA' are isosceles triangles, therefore $MA = MA'$ and $PA = PA'$. Similarly, since $[Oy$ is the mediator of the segment $[BB']$, both triangles NBB' and QBB' are isosceles triangles, therefore $NB = NB'$ and $QB = QB'$. It results that $AM + MN + NB = A'M + MN + NB' = A'B'$ and $AP + PQ + QB = A'P + PQ + QB'$. Since the shortest path between two points is a straight line, we have $A'P + PQ + QB' > A'B'$, which is equivalent to $AP + PQ + QM > AM + MN + NB$. Therefore, the points on the shores of the two rivers where John should take the pictures are M and N, the intersections of $[A'B']$ with the shores of the rivers.

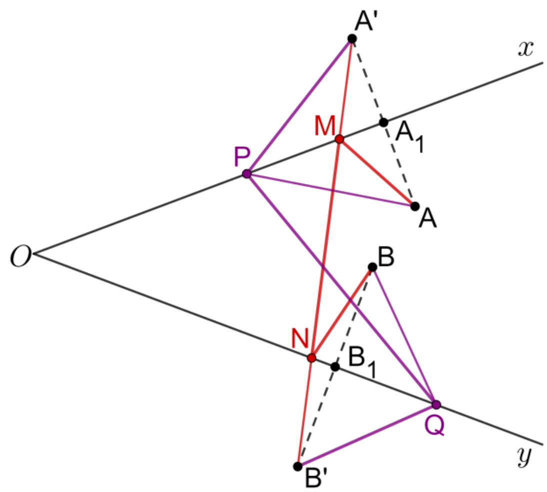

Figure 7

Problem 5. Let A and B be two points on opposite sides of a river (suppose that the river has the same width everywhere). Where should a bridge be built such that the distance from A to B is minimal? Generalization for two points A and B separated by $n \geq 1$ parallel channels.

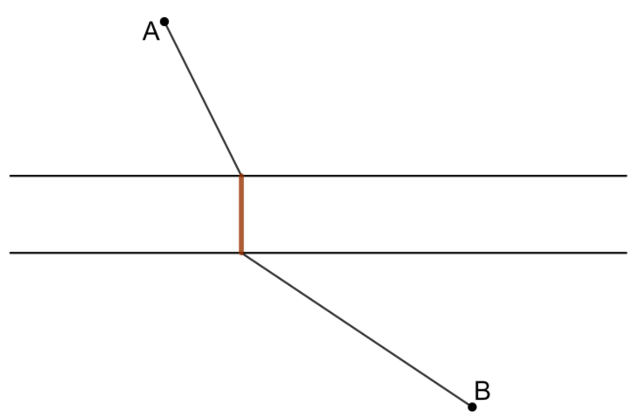

Figure 8

Solution. Let w be the width of the river and denote by s and s' the shores of the river. Because the river has the same width everywhere, $s \parallel s'$. Let $BB' \perp s$, $BB' = w$ (see Figure 9). Let $[AB'] \cap s = \{M\}$ and $MN \perp s$, $N \in s'$ (obviously, $MN = w$). Since $BB' \parallel MN$ (they are both orthogonal to s) and $BB' = MN\,(= w)$, $BB'MN$ is a parallelogram (possibly degenerate, if $AB \perp s$, which implies A, B, B' collinear and thus B, B', M, N collinear). Hence $MB' = NB$. Consequently,

$$AB' = AM + MB' = AM + NB. \qquad (1)$$

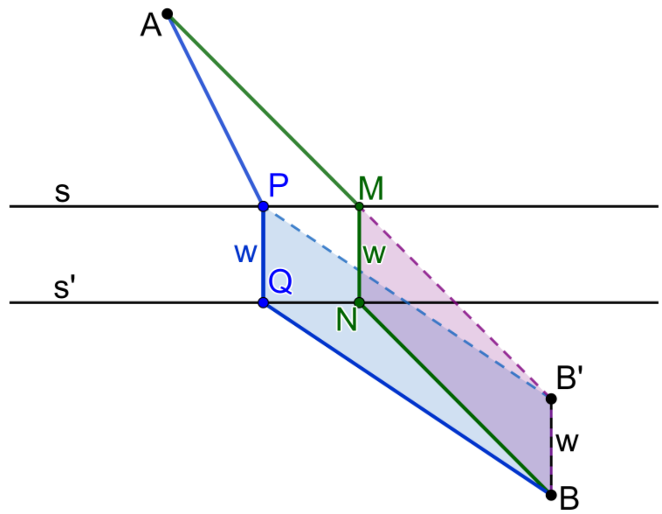

Figure 9

Now, for any point $P \in s$, $P \neq M$, if $PQ \perp s$, $Q \in s'$, then $PQ = w = BB'$. Also, $PQ \parallel BB'$ because they are both orthogonal to s. Therefore, $PQBB'$ is a parallelogram (possibly degenerate, if $PB \perp s$, which yields P, Q, B, B' collinear). Consequently, $QB = PB'$. We see that

$$AP + QB = AP + PB' > AB' = AM + NB \qquad (1)$$

By adding $PQ = MN (= w)$ to this inequality, we get $AP + PQ + QB > AM + MN + NB$, which proves that the bridge which would ensure the minimal distance between A and B is MN.

First generalization. Let A and B be separated by **two** parallel channels of widths w_1 and w_2 (Figure 10). Denote by s_i and s_i' the shores of channel i, for $i \in \overline{1,2}$. Suppose that each of the channels has the same width everywhere, hence $s_1 \parallel s_1' \parallel s_2 \parallel s_2'$.

Let $BB_1' \perp s_1$, $BB_1' = w_1 + w_2$.

Let $[AB_1'] \cap s_1 = \{M_1\}$ and $M_1N_1 \perp s_1$, $N_1 \in s_1'$ (obviously, $M_1N_1 = w_1$). Now N_1 and B are separated by **only one** channel, of width w_2. According to the first part of this proof, in order to cover the minimum distance between N_1 and B we have to build the second bridge the following way: let $BB_2' \perp s_2$, $BB_2' = w_2$ (hence $B - B_2' - B_1'$ and $B_2'B_1' = BB_1' - BB_2' = w_1$); let $[N_1B_2'] \cap s_2 = \{M_2\}$ and $M_2N_2 \perp s_2$, $N_2 \in s_2'$ (thus, $M_2N_2 = w_2$).

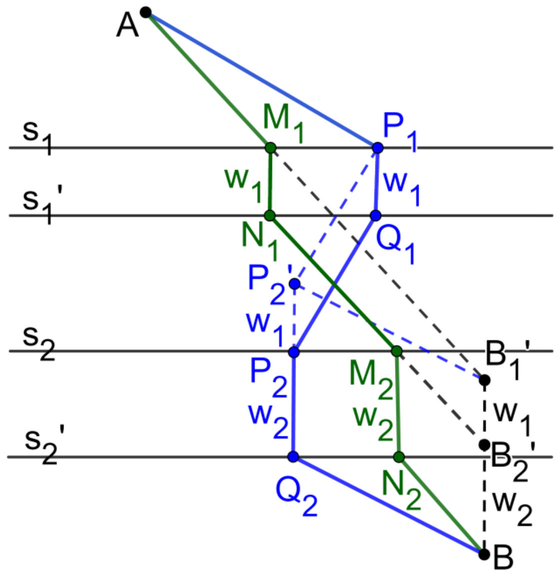

Figure 10

We claim that $AM_1 + M_1N_1 + N_1M_2 + M_2N_2 + N_2B$ is the minimal distance between A and B in the conditions of the problem. Indeed, from the first part of the proof we also know that

$$N_1B_2' = N_1M_2 + N_2B. \qquad (2)$$

On the other hand, since $B_1'B_2' = M_1N_1 (= w_1)$ and $B_1'B_2' \parallel M_1N_1$ (they are both orthogonal to the shorelines), we get that $B_1'B_2'N_1M_1$ is a parallelogram (possibly degenerate, if AB is orthogonal to the shore lines, which implies that all the points $A, B, B_1'M_1, N_1, B_2', M_2, N_2$ are collinear). Consequently, $M_1B_1' = N_1B_2'$. Therefore, relation (2) can be rewritten

$$M_1B_1' = N_1M_2 + N_2B. \qquad (3)$$

Let now $P_1Q_1 \perp s_1$, with $P_1 \in s_1$, $Q_1 \in s_1'$ (so $P_1Q_1 = w_1$) and $P_2Q_2 \perp s_2$, with $P_2 \in s_2, Q_2 \in s_2'$ (which implies $P_2Q_2 = w_2$). We want to prove that if $P_1 \neq M_1$ or $P_2 \neq M_2$, then

$$AP_1 + P_1Q_1 + Q_1P_2 + P_2Q_2 + Q_2B > AM_1 + M_1N_1 + N_1M_2 + M_2N_2 + N_2B.$$

Indeed, let $P_2' \in Q_2P_2$ such that $Q_2 - P_2 - P_2'$ and $P_2P_2' = w_1 = P_1Q_1$. Since the lines $P_2P_2'$and P_1Q_1 are also parallel (they are both orthogonal to the shore lines), the quadrilateral $P_2P_2'P_1Q_1$ is a parallelogram (possibly degenerate, if P_1P_2 is orthogonal to the shore lines). Hence, $Q_1P_2 = P_1P_2'$. On the other hand, $P_2'Q_2 = P_2'P_2 + P_2Q_2 = w_1 + w_2 = BB_1'$. Since the lines $P_2'Q_2$ and BB_1' are also parallel (they are both orthogonal to the shore lines), the quadrilateral $P_2'Q_2BB_1'$is a parallelogram (possibly degenerate, if BP_2 is orthogonal to the shore lines, which implies P_2, Q_2, P_2', B, B_1'collinear). Therefore, $Q_2B = P_2'B_1'$. Thus,

$$AP_1 + Q_1P_2 + Q_2B = AP_1 + P_1P_2' + P_2'B_1' > AB_1',$$

because the shortest path between A and B_1' is the straight line. By consequence,

$$\begin{aligned} AP_1 + P_1Q_1 + Q_1P_2 + P_2Q_2 + Q_2B &= AP_1 + w_1 + Q_1P_2 + w_2 + Q_2B > w_1 + w_2 + AB_1' = \\ &= w_1 + w_2 + AM_1 + M_1B_1' \overset{(3)}{=} w_1 + w_2 + AM_1 + N_1M_2 + N_2B = \\ &= AM_1 + M_1N_1 + N_1M_2 + M_2N_2 + N_2B, \end{aligned}$$

that completes the proof.

Summarizing, if A and B are separated by two channels, then for a minimal distance we must build the bridges the following way: we take $BB_1' \perp s_1$, $BB_1' = w_1 + w_2$, then $[AB_1'] \cap s_1 = \{M_1\}$ and $M_1N_1 \perp s_1$, $N_1 \in s_1'$; in the second step we take $B_2' \in (BB_1')$ such that $BB_2' = w_2$,then $[N_1B_2'] \cap s_2 = \{M_2\}$ and finally $M_2N_2 \perp s_2$, $N_2 \in s_2'$.

Second generalization. Let A and B be separated by n parallel channels of widths $w_1, w_2, \ldots, w_n$, respectively. Denote by s_i and s_i' the shores of channel i, for $i \in \overline{1,n}$. Suppose that each of the channels has the same width everywhere, hence $s_1 \parallel s_1' \parallel s_2 \parallel s_2' \parallel \cdots \parallel s_n \parallel s_n'$. We want to determine the position of n bridges over the n channels in order to get the shortest road from A to B by crossing these bridges.

It can be proved by mathematical induction that the bridges must be built the following way:

- in the first step we take $BB_1' \perp s_1$, $BB_1' = \sum_{i=1}^{n} w_i$, then $[AB_1'] \cap s_1 = \{M_1\}$ and $M_1N_1 \perp s_1$, $N_1 \in s_1'$;

- in the second step we take $B_2' \in (BB_1')$ such that $BB_2' = \sum_{i=2}^{n} w_i$,then $[N_1B_2'] \cap s_2 = \{M_2\}$ and $M_2N_2 \perp s_2$, $N_2 \in s_2'$;

$\vdots$

- generally, in step $k \in \overline{2,n}$ we take $B_k' \in (BB_1')$ such that $BB_k' = \sum_{i=k}^{n} w_i$,then $[N_{k-1}B_k'] \cap s_k = \{M_k\}$ and $M_kN_k \perp s_k$, $N_k \in s_k'$.

The proof uses the same ideas as the first generalization (case $n=2$). Basically, if we compare the path $AM_1N_1M_2N_2\ldots M_nN_nB$ with any other path $AP_1Q_1P_2Q_2\ldots P_nQ_nB$, they have in common the lengths of the bridges ($M_iN_i = P_iQ_i = w_i$, for $i \in \overline{1,n}$). What distinguishes these two paths is that, using convenient parallelograms, the remaining segments of the first path can be moved, side by side, to cover the segment AB_1', while the remaining segments of the second path will move on a *broken line* uniting the points A and B_1' (in Figure 11 we represented the case $n=4$, for a better understanding). Therefore, the first path is the shortest.

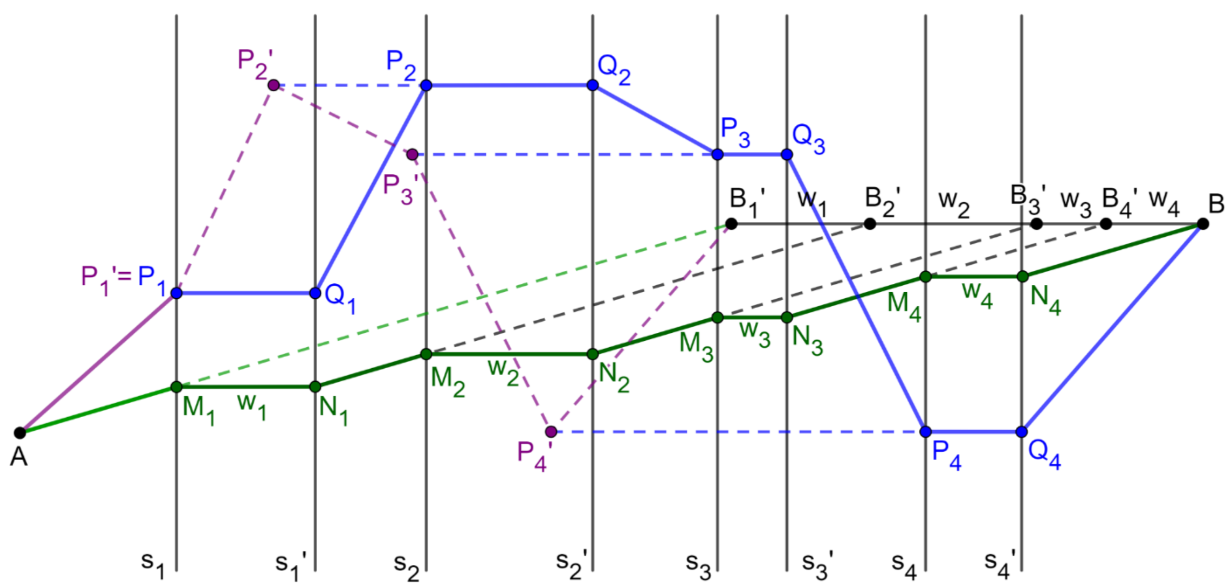

Figure 11

4 CONCLUSION

We have used Calculus to answer the questions in the research proposal. Also, we have written a C++ code to approximate the solutions of the equations obtained and to observe simpler particular cases. We expect that if the speed with which the road is traveled varies in a more complicated way, the derivation of the equations will not be easy at all.

In the sequel we considered three related problems. By considering the speed of the traveler constant, we were able to use a geometrical approach to solve them.

EDITION NOTES

[1] We have all the information that we need about the function $f'(x)$. Then, it would have been simpler to write a program that approximates the solution of $f'(x) = 0$. For instance, we can develop a program that determines a sequence $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots$ of intervals such that $f'(a_i) < 0$ and $f'(b_i) > 0$, and terminates when $b_i - a_i$ is less than a given value. A program of this kind is also likely to be computationally more efficient than that considered in the work.